

# DEGENERATIONS OF THE FRÖLICHER SPECTRAL SEQUENCES OF SOLVMANIFOLDS

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**ABSTRACT.** We show that the Frölicher spectral sequence of a complex parallelizable solvmanifold is degenerate at  $E_2$  term. For a semi-direct product  $G = \mathbb{C}^n \ltimes_{\phi} N$  of Lie groups with lattice  $\Gamma = \Gamma' \ltimes \Gamma''$  such that  $N$  is a nilpotent Lie group with a left-invariant complex structure and  $\phi$  is a semi-simple action, we also show that, if the Frölicher spectral sequence of the nilmanifold  $N/\Gamma''$  is degenerate at  $E_r$  term for  $r \geq 2$ , then the spectral sequence of the solvmanifold  $G/\Gamma$  is also degenerate at  $E_r$  term.

## 1. MAIN RESULT

Let  $M$  be a compact complex manifold. The Frölicher spectral sequence  $E_{*,*}^*(M)$  of  $M$  is the spectral sequence of the  $\mathbb{C}$ -valued de Rham complex  $(A^*(M), d)$  which is given by the double complex  $(A^{*,*}(M), \partial, \bar{\partial})$ . Consider the differential  $d_*$  on  $E_{*,*}^*(M)$ . We denote

$$r(M) = \min\{r \in \mathbb{N} \mid \forall s \geq r, d_s = 0\}.$$

It is well-known that  $r(M) = 1$  if  $M$  is Kähler. In general  $r(M) = 1$  does not hold. Hence we can say that  $r(M)$  measures how far  $M$  is from being a Kähler manifold. We are interested in finding non-Kähler complex manifolds which has large  $r(M)$ .

Consider a nilmanifold  $G/\Gamma$  with a left-invariant complex structure where  $G$  is a simply connected nilpotent Lie group and  $\Gamma$  is a lattice in  $G$ . In [1], it is proved that if  $G/\Gamma$  is complex parallelizable then  $r(G/\Gamma) \leq 2$ . However in general for arbitrary  $k \in \mathbb{N}$  Rollenske proved that there exists a nilmanifold  $G/\Gamma$  with a left-invariant complex structure such that  $r(G/\Gamma) \geq k$ .

Considering solvmanifolds, it is natural to expect that we can find wider variety of complex solvmanifolds with large  $r(M)$  than the case of nilmanifolds. In this paper we consider such expectation.

We extend the result in [1]. We prove:

**Theorem 1.1.** *Let  $G$  be a simply connected complex solvable Lie group with a lattice  $\Gamma$ . Then we have  $r(G/\Gamma) \leq 2$ .*

We consider a solvable Lie group  $G$  with the following assumption.

**Assumption 1.2.**  *$G$  is the semi-direct product  $\mathbb{C}^n \ltimes_{\phi} N$  so that:*

- (1)  *$N$  is a simply connected nilpotent Lie group with a left-invariant complex structure  $J$ . Let  $\mathfrak{a}$  and  $\mathfrak{n}$  be the Lie algebras of  $\mathbb{C}^n$  and  $N$  respectively.*
- (2) *For any  $t \in \mathbb{C}^n$ ,  $\phi(t)$  is a holomorphic automorphism of  $(N, J)$ .*
- (3)  *$\phi$  induces a semi-simple action on the Lie algebra  $\mathfrak{n}$  of  $N$ .*
- (4)  *$G$  has a lattice  $\Gamma$ . (Then  $\Gamma$  can be written by  $\Gamma = \Gamma' \ltimes_{\phi} \Gamma''$  such that  $\Gamma'$  and  $\Gamma''$  are lattices*

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of  $\mathbb{C}^n$  and  $N$  respectively and for any  $t \in \Gamma'$  the action  $\phi(t)$  preserves  $\Gamma''$ .)  
 (5) The inclusion  $\bigwedge^{*,*} \mathfrak{n}^* \subset A^{*,*}(N/\Gamma'')$  induces an isomorphism

$$H_{\bar{\partial}}^{*,*}(\mathfrak{n}) \cong H_{\bar{\partial}}^{*,*}(N/\Gamma'').$$

Then in [4], we construct an explicit finite dimensional sub-DBA  $B^{*,*} \subset (A^{*,*}(G/\Gamma), \bar{\partial})$  which compute the Dolbeault cohomology  $H_{\bar{\partial}}^{*,*}(G/\Gamma)$  of  $G/\Gamma$ . By this, we can observe that the Dolbeault cohomology  $H_{\bar{\partial}}^{*,*}(G/\Gamma)$  varies for a choice of a lattice  $\Gamma$ . By this, the computation of  $H_{\bar{\partial}}^{*,*}(G/\Gamma)$  is more complicated than the computation of  $H_{\bar{\partial}}^{*,*}(N/\Gamma'')$ . We prove:

**Theorem 1.3.** *Let  $G$  be a Lie group as in Assumption 1.2. Then we have:*

- If  $r(N/\Gamma'') = 1$ , then we have  $r(G/\Gamma) \leq 2$ .
- If  $r(N/\Gamma'') > 1$ , then we have  $r(G/\Gamma) \leq r(N/\Gamma'')$ .

**Corollary 1.4.** *Let  $G$  be a Lie group as in Assumption 1.2. Suppose  $N$  is complex parallelizable. Then we have  $r(G/\Gamma) \leq 2$ .*

## 2. PRELIMINARLY: FINITE DIMENSIONAL DGAs OF PD-TYPE

In this section we study the homological algebra of finite dimensional DGA like the theory of harmonic forms on compact Hermitian manifolds.

**Definition 2.1.** (DGA) A differential graded algebra (DGA) is a graded commutative  $\mathbb{C}$ -algebra  $A^*$  with a differential  $d$  of degree  $+1$  so that  $d \circ d = 0$  and  $d(\alpha \cdot \beta) = d\alpha \cdot \beta + (-1)^p \alpha \cdot d\beta$  for  $\alpha \in A^p$ .

(DBA) A differential bigraded algebra (DBA) is a DGA  $(A^*, \bar{\partial})$  such that  $A^*$  is bigraded as  $A^r = \bigoplus_{r=p+q} A^{p,q}$  and the differential  $\bar{\partial}$  has type  $(0, 1)$ .

(BBA) A bidifferential bigraded algebra (BBA) is a DBA  $(A^*, \bar{\partial})$  with another differential  $\partial$  of type  $(1, 0)$  such that  $\partial\bar{\partial} + \bar{\partial}\partial = 0$ .

Let  $A^*$  be a finite dimensional graded commutative  $\mathbb{C}$ -algebra.

**Definition 2.2.**  $A^*$  is of PD-type if the following conditions hold:

- $A^{* < 0} = 0$  and  $A^0 = \mathbb{C}1$  where  $1$  is the identity element of  $A^*$ .
- For some positive integer  $n$ ,  $A^{* > n} = 0$  and  $A^n = \mathbb{C}v$  for  $v \neq 0$ .
- For any  $0 < i < n$  the bi-linear map  $A^i \times A^{n-i} \ni (\alpha, \beta) \mapsto \alpha \cdot \beta \in A^n$  is non-degenerate.

Suppose  $A^*$  is of PD-type. Let  $h$  be a Hermitian metric on  $A^*$  which is compatible with the grading. Take  $v \in A^n$  such that  $h(v, v) = 1$ . Define the  $\mathbb{C}$ -anti-linear map  $\bar{*} : A^i \rightarrow A^{n-i}$  as  $\alpha \cdot \bar{*}\beta = h(\alpha, \beta)v$ .

**Definition 2.3.** A finite dimensional DGA  $(A^*, d)$  is of PD-type if the following conditions hold:

- $A^*$  is a finite dimensional graded  $\mathbb{C}$ -algebra of PD-type.
- $dA^{n-1} = 0$  and  $dA^0 = 0$ .

Let  $(A^*, d)$  be a finite dimensional DGA of PD-type. Denote  $d^* = -\bar{*}d\bar{*}$ .

**Lemma 2.4.** *We have  $h(d\alpha, \beta) = h(\alpha, d^*\beta)$  for  $\alpha \in A^{i-1}$  and  $\beta \in A^i$ .*

*Proof.* By  $dA^{n-1} = 0$ , we have

$$d\alpha \cdot \bar{*}\beta = d(\alpha \cdot \bar{*}\beta) - (-1)^{i-1} \alpha \cdot (d\bar{*}\beta) = (-1)^p \alpha \cdot (d\bar{*}\beta) = \alpha \cdot (\bar{*}d\bar{*}\beta).$$

Hence we have

$$h(d\alpha, \beta)v = h(\alpha, d^*\beta)v.$$

□

Define  $\Delta = dd^* + d^*d$ . and  $\mathcal{H}^*(A) = \ker \Delta$ . By Lemma 2.4 and finiteness of the dimension of  $A^*$ , we can easily show the following lemma like the proof of [10, Theorem 5.23].

**Lemma 2.5.** *We have the decomposition*

$$A^p = \mathcal{H}^p(A) \oplus \Delta(A^p) = \mathcal{H}^p(A) \oplus d(A^{p-1}) \oplus d^*(A^{p+1}).$$

*By this decomposition, the inclusion  $\mathcal{H}^*(A) \subset A^*$  induces a isomorphism*

$$\mathcal{H}^p(A) \cong H^p(A)$$

*of vector spaces.*

**Lemma 2.6.** *Let  $(A^*, d)$  be a finite dimensional DGA of PD-type. Then the cohomology algebra  $H^*(A)$  is a finite dimensional graded commutative  $\mathbb{C}$ -algebra of PD-type.*

*Proof.* Since the restriction  $\bar{*} : \mathcal{H}^i(A) \rightarrow \mathcal{H}^{n-i}(A)$  is also an isomorphism, the linear map  $\mathcal{H}^i(A) \times \mathcal{H}^{n-i}(A) \ni (\alpha, \beta) \rightarrow \alpha \cdot \beta \in \mathcal{H}^n(A) = A^n$  is non-degenerate. Hence the lemma follows from Lemma 2.5.  $\square$

**Lemma 2.7.** *Let  $(A^*, d)$  be a finite dimensional DGA of PD-type and  $B^* \subset A^*$  be a sub-DGA of PD-type. Then the inclusion  $B^* \subset A^*$  induces an injection*

$$H^*(B^*) \hookrightarrow H^*(A^*).$$

*Proof.* By the inclusion  $\mathcal{H}^*(B) \subset \mathcal{H}^*(A)$  the Lemma follows from Lemma 2.5.  $\square$

**Proposition 2.8.** *Let  $(A^{*,*}, d = d' + d'')$  be a finite dimensional BBA such that  $(\text{Tot} A^{*,*}, d' + d'')$  is a finite dimensional DGA of PD-type. Let  $B^{*,*} \subset A^{*,*}$  be a sub-BBA such that  $(\text{Tot} B^{*,*}, d' + d'')$  is a finite dimensional DGA of PD-type. Consider the spectral sequences  $E_r^{*,*}(A^{*,*})$  and  $E_r^{*,*}(B^{*,*})$  given by the BBA-structure. Then for each  $r$ ,  $\text{Tot} E_r^{*,*}(A)$  and  $\text{Tot} E_r^{*,*}(B)$  are finite dimensional DGAs of PD-type and the inclusion  $B^{*,*} \subset A^{*,*}$  induces an injection  $E_r^{*,*}(A^{*,*}) \hookrightarrow E_r^{*,*}(B^{*,*})$ .*

*Proof.* We will prove the proposition inductively. By the assumption  $\text{Tot} E_0^{*,*}(A^{*,*}) \cong (\text{Tot} A^{*,*}, d'')$  and  $\text{Tot} E_0^{*,*}(B^{*,*}) \cong (\text{Tot} B^{*,*}, d'')$  are finite dimensional DGAs of PD-type. Suppose that for some  $r$   $\text{Tot} E_r^{*,*}(A)$  and  $\text{Tot} E_r^{*,*}(B)$  are finite dimensional DGAs of PD-type and the inclusion  $B^{*,*} \subset A^{*,*}$  induces an injection  $E_r^{*,*}(A^{*,*}) \hookrightarrow E_r^{*,*}(B^{*,*})$ . Since we have  $H^*(\text{Tot} E_r^{*,*}(A)) \cong \text{Tot} E_{r+1}^{*,*}(A)$  and  $H^*(\text{Tot} E_r^{*,*}(B)) \cong \text{Tot} E_{r+1}^{*,*}(B)$ , by Lemma 2.6 and 2.7,  $\text{Tot} E_{r+1}^{*,*}(A)$  and  $\text{Tot} E_{r+1}^{*,*}(B)$  are finite dimensional DGAs of PD-type and the induced map  $E_{r+1}^{*,*}(A^{*,*}) \rightarrow E_{r+1}^{*,*}(B^{*,*})$  is injective.  $\square$

### 3. PROOF OF THEOREM 1.1

Let  $G$  be a simply connected solvable Lie group. Denote by  $\mathfrak{g}_+$  (resp.  $\mathfrak{g}_-$ ) the Lie algebra of the left-invariant holomorphic (anti-holomorphic) vector fields on  $G$ . As a Lie algebra we have an isomorphism  $\mathfrak{g}_+ \cong \mathfrak{g}_-$  by the complex conjugate. Let  $N$  be the nilradical of  $G$ . We can take a simply connected complex nilpotent subgroup  $C \subset G$  such that  $G = C \cdot N$  (see [3]). Since  $C$  is nilpotent, the map

$$C \ni c \mapsto (\text{Ad}_c)_s \in \text{Aut}(\mathfrak{g}_+)$$

is a homomorphism where  $(\text{Ad}_c)_s$  is the semi-simple part of  $\text{Ad}_s$ . Denote by  $\bigwedge \mathfrak{g}_+^*$  (resp.  $\bigwedge \mathfrak{g}_-^*$ ) the sub-DGA of  $(A^{*,0}, \partial)$  (resp.  $(A^{0,*}, \bar{\partial})$ ) which consists of the left-invariant holomorphic (anti-holomorphic) forms. As a DGA, we have an isomorphism between  $(\bigwedge \mathfrak{g}_+^*, \partial)$  and  $(\bigwedge \mathfrak{g}_-^*, \bar{\partial})$  given by the complex conjugate.

We have a basis  $X_1, \dots, X_n$  of  $\mathfrak{g}_+$  such that  $(\text{Ad}_c)_s = \text{diag}(\alpha_1(c), \dots, \alpha_n(c))$  for  $c \in C$ . Let  $x_1, \dots, x_n$  be the basis of  $\mathfrak{g}_+^*$  which is dual to  $X_1, \dots, X_n$ . For a multi-index  $I = \{i_1, \dots, i_p\}$  we write  $x_I = x_{i_1} \wedge \dots \wedge x_{i_p}$ , and  $\alpha_I = \alpha_{i_1} \dots \alpha_{i_p}$ .

**Theorem 3.1.** ([7, Corollary 6.2 and its proof]) *Let  $B_\Gamma^*$  be the subcomplex of  $(A^{0,*}(G/\Gamma), \bar{\partial})$  defined as*

$$B_\Gamma^* = \left\langle \frac{\bar{\alpha}_I}{\alpha_I} \bar{x}_I \middle| \left( \frac{\bar{\alpha}_I}{\alpha_I} \right)_{|\Gamma} = 1 \right\rangle.$$

*Then the inclusion  $B_\Gamma^* \subset A^{0,*}(G/\Gamma)$  induces an isomorphism*

$$H^*(B_\Gamma^*) \cong H^{0,*}(G/\Gamma).$$

By this theorem, since  $G/\Gamma$  is complex parallelizable, for the DBA  $(\bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^*, \bar{\partial})$ , the inclusion  $\bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^* \subset A^{*,*}$  induce an isomorphism

$$\bigwedge \mathfrak{g}_+^* \otimes H^*(B_\Gamma^*) \cong H^{*,*}(G/\Gamma).$$

We consider the weight decomposition

$$\bigwedge \mathfrak{g}_+^* = \bigoplus V_{\nu_k}$$

of the action

$$C \ni c \mapsto (\text{Ad}_c)_s \in \text{Aut}(\bigwedge \mathfrak{g}_+^*).$$

Then we have

$$B_\Gamma^* = \bigoplus_{\left(\frac{\nu_k}{\bar{\nu}_k}\right)_{|\Gamma} = 1} \frac{\nu_k}{\bar{\nu}_k} V_{\bar{\nu}_k}$$

and hence

$$\bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^* = \bigoplus_{\left(\frac{\nu_k}{\bar{\nu}_k}\right)_{|\Gamma} = 1} (\nu_k \bigwedge \mathfrak{g}_+^*) \otimes \left(\frac{1}{\bar{\nu}_k} V_{\bar{\nu}_k}\right).$$

Regard  $\bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^*$  as a BBA  $(\bigwedge \mathfrak{g}_+^* \otimes B_\Gamma^*, \partial, \bar{\partial})$ . Then for each  $\nu_k$  the double complex  $(\nu_k \bigwedge \mathfrak{g}_+^*) \otimes (\frac{1}{\bar{\nu}_k} V_{\bar{\nu}_k})$  is the tensor product of  $(\nu_k \bigwedge \mathfrak{g}_+^*, \partial)$  and  $(\frac{1}{\bar{\nu}_k} V_{\bar{\nu}_k}, \bar{\partial})$ . By the Künneth theorem, the spectral sequence of the double complex of a tensor product of two cochain complexes is degenerate at  $E_2$  term. Hence Theorem 1.1 follows.

#### 4. PROOF OF THEOREM 1.3

Let  $G$  be a Lie group as in Assumption 1.2. Consider the decomposition  $\mathfrak{n}_\mathbb{C} = \mathfrak{n}^{1,0} \oplus \mathfrak{n}^{0,1}$ . By the condition (2), this decomposition is a direct sum of  $\mathbb{C}^n$ -modules. By the condition (3) we have a basis  $Y_1, \dots, Y_m$  of  $\mathfrak{n}^{1,0}$  such that the action  $\phi$  on  $\mathfrak{n}^{1,0}$  is represented by  $\phi(t) = \text{diag}(\alpha_1(t), \dots, \alpha_m(t))$ . Since  $Y_j$  is a left-invariant vector field on  $N$ , the vector field  $\alpha_j Y_j$  on  $\mathbb{C}^n \times_\phi N$  is left-invariant. Hence we have a basis  $X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m$  of  $\mathfrak{g}^{1,0}$ . Let  $x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m$  be the basis of  $\bigwedge^{1,0} \mathfrak{g}^*$  which is dual to  $X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m$ . Then we have

$$\bigwedge \mathfrak{g}^* = \bigwedge^{p,q} \langle x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m \rangle \otimes \bigwedge^q \langle \bar{x}_1, \dots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1, \dots, \bar{\alpha}_m^{-1} \bar{y}_m \rangle.$$

**Lemma 4.1.** ([4]) *There exist the unique unitary characters  $\beta_i$  and  $\gamma_i$  on  $\mathbb{C}^n$  such that  $\alpha_i \beta_i^{-1}$  and  $\bar{\alpha} \gamma_i^{-1}$  are holomorphic.*

**Theorem 4.2.** ([4, Corollary 4.2]) *Let  $B^{*,*} \subset A^{*,*}(G/\Gamma)$  be the subDBA of  $A^{*,*}(G/\Gamma)$  given by*

$$B^{p,q} = \left\langle x_I \wedge \alpha_J^{-1} \beta_{JY_J} \wedge \bar{x}_K \wedge \bar{\alpha}_L^{-1} \gamma_L \bar{y}_L \mid \begin{array}{l} |I| + |K| = p, |J| + |L| = q \\ \text{the restriction of } \beta_J \gamma_L \text{ on } \Gamma \text{ is trivial} \end{array} \right\rangle.$$

*Then the inclusion  $B^{*,*} \subset A^{*,*}(G/\Gamma)$  induces a cohomology isomorphism*

$$H_{\bar{\partial}}^{*,*}(B^{*,*}) \cong H_{\bar{\partial}}^{*,*}(G/\Gamma).$$

*Remark 1.* By the construction  $(B^{*,*}, \bar{\partial})$  is a sub-DBA of the DBA  $(\bigwedge^{*,*}(\mathbb{C}^n \oplus \mathfrak{n}_{\mathbb{C}})^*, \bar{\partial})$ . However, considering the differential  $\partial$ ,  $(B^{*,*}, \partial, \bar{\partial})$  is not a sub-BBA of the BBA  $(\bigwedge^{*,*}(\mathbb{C}^n \oplus \mathfrak{n}_{\mathbb{C}})^*, \partial, \bar{\partial})$

The action  $\phi$  induces the action of  $\mathbb{C}^n$  on the BBA  $(\bigwedge^{*,*} \mathfrak{n}_{\mathbb{C}}^*, \bar{\partial}, \bar{\partial})$ . We consider the weight decomposition

$$\bigwedge^{*,*} \mathfrak{n}_{\mathbb{C}}^* = \bigoplus_{\mu_i} A_{\mu_i}$$

of this action. Then each  $A_{\mu_i}$  is a sub-Double-complex of  $\bigwedge^{*,*} \mathfrak{n}_{\mathbb{C}}^*$ . Let  $B^{*,*}$  be the DBA as in Theorem 4.2. Then we have

$$B^{*,*} = \bigoplus_{\mu_i} \bigwedge \mathbb{C}^n \otimes (\lambda_i \mu_i^{-1} A_{\mu_i})$$

where  $\lambda_i$  are unique unitary characters such that  $\lambda_i \mu_i^{-1}$  are holomorphic. If  $(\mu_i)_{|\Gamma} = 1$ , then  $\mu_i$  is unitary and so  $\lambda_i = \mu_i$ . If  $(\mu_i)_{|\Gamma} \neq 1$  and  $(\lambda_i)_{|\Gamma} = 1$ , then  $\lambda_i \mu_i^{-1}$  is a non-trivial holomorphic character. Consider the double-complexes

$$C^{*,*} = \bigoplus_{(\mu_i)_{|\Gamma} = 1} \bigwedge \mathbb{C}^n \otimes A_{\mu_i}$$

and

$$D^{*,*} = \bigoplus_{(\mu_i)_{|\Gamma} \neq 1, (\lambda_i)_{|\Gamma} = 1} (\lambda_i \mu_i^{-1} \bigwedge \mathbb{C}^n) \otimes A_{\mu_i}.$$

Then we have

$$B^{*,*} = C^{*,*} \oplus D^{*,*}.$$

Consider the spectral sequences  $E_*^{*,*}(B)$ ,  $E_*^{*,*}(C)$  and  $E_*^{*,*}(D)$  of the double complexes  $B^{*,*}$ ,  $C^{*,*}$  and  $D^{*,*}$  respectively. Then we have

$$E_2^{*,*}(D) = \bigoplus_{(\mu_i)_{|\Gamma} \neq 1, (\lambda_i)_{|\Gamma} = 1} H_{\partial}^{*,*}((\lambda_i \mu_i^{-1} \bigwedge \mathbb{C}^n) \otimes H_{\bar{\partial}}^{*,*}(A_{\mu_i})).$$

Since  $\lambda_i \mu_i^{-1}$  is a non-trivial holomorphic character, we have  $H_{\partial}^{*,*}(\lambda_i \mu_i^{-1} \bigwedge \mathbb{C}^n) = 0$  and so we have

$$E_2^{*,*}(D) = \bigoplus_{(\mu_i)_{|\Gamma} \neq 1, (\lambda_i)_{|\Gamma} = 1} H_{\partial}^{*,*}((\lambda_i \mu_i^{-1} \bigwedge \mathbb{C}^n) \otimes H_{\bar{\partial}}^{*,*}(A_{\mu_i})) = 0.$$

Hence we have:

**Lemma 4.3.**

$$E_2^{*,*}(B) = E_2^{*,*}(C).$$

*Remark 2.* By this lemma and the property of spectral sequence (see [8, Theorem 3.5]), we have an isomorphism  $H^*(\text{Tot} C^{*,*}) \cong H^*(G/\Gamma)$ . This isomorphism also follows from the result in [5].

Now we have  $C^{*,*} \subset \bigwedge(\mathbb{C}^n \oplus \mathfrak{n})_{\mathbb{C}}^*$ . Since we have

$$C^{p,q} = \left\langle x_I \wedge y_J \wedge \bar{x}_K \wedge \bar{y}_L \mid \begin{array}{l} |I| + |K| = p, |J| + |L| = q \\ \text{the restriction of } \alpha_J^{-1} \bar{\alpha}_L^{-1} \text{ on } \Gamma \text{ is trivial} \end{array} \right\rangle,$$

$C^{*,*}$  is closed under wedge product and we have  $\bar{*}(C^{*,*}) \subset C^{*,*}$  where  $\bar{*}$  is the Hodge star operator of the left-invariant Hermitian metric

$$x_1 \bar{x}_1 + \cdots + x_n \bar{x}_n + y_1 \bar{y}_1 + \cdots + y_m \bar{y}_m$$

on  $\mathbb{C}^n \times N$ . Hence  $\text{Tot} C^{*,*}$  is a finite dimensional DGA of PD-type. By Proposition 2.8, we have an inclusion  $E_r(C) \hookrightarrow E_r(\bigwedge(\mathbb{C}^n \oplus \mathfrak{n})_{\mathbb{C}}^*)$ . By Theorem 4.2 and 4.3, we have

$$E_r(G/\Gamma) \cong E_r(B) = E_r(C)$$

for  $r \geq 2$ . Hence we have an injection

$$E_r(G/\Gamma) \hookrightarrow E_r(\bigwedge(\mathbb{C}^n \oplus \mathfrak{n})_{\mathbb{C}}^*).$$

This implies the Theorem.

## 5. EXAMPLES AND REMARKS

In this section we notice:

*Remark 3.* Consider a Lie group as Assumption 1.2.

- There exists an example  $G/\Gamma$  such that  $r(G/\Gamma) = 2$  but  $N/\Gamma''$
- There exists an example  $G/\Gamma$  such that  $r(G/\Gamma) < r(N/\Gamma'')$ .

*Example 1.* Let  $G = \mathbb{C} \ltimes_{\phi} \mathbb{C}^2$  such that  $\phi(x + \sqrt{-1}y) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$ . Then for some  $a \in \mathbb{R}$  the matrix  $\begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}$  is conjugate to an element of  $SL(2, \mathbb{Z})$ . Hence for any  $0 \neq b \in \mathbb{R}$  we have a lattice  $\Gamma = (a\mathbb{Z} + b\sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$  such that  $\Gamma''$  is a lattice of  $\mathbb{C}^2$ .

If  $b = 2\pi$ , then the Dolbeault cohomology  $H^{*,*}(G/\Gamma)$  is isomorphic to the Dolbeault cohomology of the complex 3-torus (see [4]). In this case we have

$$\dim H^*(G/\Gamma) = 2 < 6 = \dim H_{\bar{\partial}}^{1,0}(G/\Gamma) + \dim H_{\bar{\partial}}^{1,0}(G/\Gamma).$$

This implies  $r(G/\Gamma) > 1$ . Hence the first assertion of Remark 3 follows.

*Remark 4.* In [6], in case  $N$  is abelian we give a condition for  $r(G/\Gamma) = 1$ .

*Example 2.* Let  $G = \mathbb{C} \ltimes_{\phi} N$  such that

$$N = \left\{ \begin{pmatrix} 1 & \bar{z} & \frac{1}{2}\bar{z}^2 & w \\ 0 & 1 & \bar{z} & v \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : z, v, w \in \mathbb{C} \right\}$$

and  $\phi$  is given by

$$\phi(s + \sqrt{-1}t) \begin{pmatrix} 1 & \bar{z} & \frac{1}{2}\bar{z}^2 & w \\ 0 & 1 & \bar{z} & v \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{-\pi\sqrt{-1}t}\bar{z} & \frac{1}{2}e^{-2\pi\sqrt{-1}t}\bar{z}^2 & e^{-\pi\sqrt{-1}t}w \\ 0 & 1 & e^{-\pi\sqrt{-1}t}\bar{z} & v \\ 0 & 0 & 1 & e^{\pi\sqrt{-1}t}z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have a lattice

$$\Gamma = (\mathbb{Z} + \sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$$

such that

$$\Gamma'' = \left\{ \begin{pmatrix} 1 & \bar{z} & \frac{1}{2}\bar{z}^2 & w \\ 0 & 1 & \bar{z} & v \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} : z, v, w \in (\mathbb{Z} + \sqrt{-1}\mathbb{Z}) \right\}.$$

Then  $G$  is a Lie group as Assumption 1.2 and we have  $x_1 = ds + \sqrt{-1}t$ ,  $y_1 = dz$ ,  $y_2 = dv - \bar{z}dz$  and  $y_3 = dw + \bar{z}dv - \frac{1}{2}\bar{z}^2dz$  where  $x_i, y_i$  are as Section 4. We have  $dy_1 = 0$ ,  $dy_2 = y_1 \wedge \bar{y}_1$ , and  $dy_3 = \bar{y}_1 \wedge y_2$ . It is known that  $r(N/\Gamma'') \geq 3$  (see [2, Example 1]). The second assertion of Remark 3 follows from the following proposition.

*Proposition 5.1.* *We have  $r(G/\Gamma) = 1$ .*

*Proof.* Since  $G/\Gamma$  is real parallelizable, we have

$$\sum_{k=0}^8 (-1)^k \dim H^*(G/\Gamma) = 0$$

and

$$\sum_{(p,q)=(0,0)}^{(4,4)} (-1)^{p,q} \dim H_{\bar{\partial}}^{p,q}(G/\Gamma) = 0$$

by the Hirzebruch–Riemann–Roch theorem. Hence by the Poincaré Duality and Serre Duality, it is sufficient to show the equations

$$\dim H_{\bar{\partial}}^{1,0}(G/\Gamma) + \dim H_{\bar{\partial}}^{0,1}(G/\Gamma) = \dim H^1(G/\Gamma),$$

$$\dim H_{\bar{\partial}}^{2,0}(G/\Gamma) + \dim H_{\bar{\partial}}^{1,1}(G/\Gamma) + \dim H_{\bar{\partial}}^{0,2}(G/\Gamma) = \dim H^2(G/\Gamma),$$

$$\dim H_{\bar{\partial}}^{3,0}(G/\Gamma) + \dim H_{\bar{\partial}}^{2,1}(G/\Gamma) + \dim H_{\bar{\partial}}^{1,2}(G/\Gamma) + \dim H_{\bar{\partial}}^{0,3}(G/\Gamma) = \dim H^3(G/\Gamma).$$

We consider  $B^{*,*}$  as Theorem 4.2. We have an isomorphism  $H_{\bar{\partial}}^{*,*}(G/\Gamma) \cong H_{\bar{\partial}}^{*,*}(B^{*,*})$ . We also have  $H^*(G/\Gamma) \cong H^*(\text{Tot} B^{*,*})$ . Then we have

$$B^{1,0} = \langle x_1, y_2 \rangle, \quad B^{0,1} = \langle \bar{x}_1, \bar{y}_2 \rangle,$$

$$B^{2,0} = \langle x_1 \wedge y_2, y_1 \wedge y_3 \rangle, \quad B^{0,2} = \langle \bar{x}_1 \wedge \bar{y}_2, \bar{y}_1 \wedge \bar{y}_3 \rangle,$$

$$B^{1,1} = \langle x_1 \wedge \bar{x}_1, x_1 \wedge \bar{y}_2, y_1 \wedge \bar{y}_1, y_1 \wedge \bar{y}_3, y_2 \wedge \bar{x}_1, y_2 \wedge \bar{y}_2, y_3 \wedge \bar{y}_1, y_3 \wedge \bar{y}_3 \rangle,$$

$$B^{3,0} = \langle x_1 \wedge y_1 \wedge y_3, y_1 \wedge y_2 \wedge y_3 \rangle, \quad B^{0,3} = \langle \bar{x}_1 \wedge \bar{y}_1 \wedge \bar{y}_3, \bar{y}_1 \wedge \bar{y}_2 \wedge \bar{y}_3 \rangle,$$

$$B^{2,1} = \langle x_1 \wedge y_1 \wedge \bar{y}_1, x_1 \wedge y_1 \wedge \bar{y}_3, x_1 \wedge y_2 \wedge \bar{x}_1, x_1 \wedge y_2 \wedge \bar{y}_2, x_1 \wedge y_3 \wedge \bar{y}_1, x_1 \wedge y_3 \wedge \bar{y}_3, y_1 \wedge y_2 \wedge \bar{y}_1, y_1 \wedge y_2 \wedge \bar{y}_3, y_1 \wedge y_3 \wedge \bar{x}_1, y_1 \wedge y_3 \wedge \bar{y}_2, y_2 \wedge y_3 \wedge \bar{y}_1, y_2 \wedge y_3 \wedge \bar{y}_3 \rangle,$$

$$B^{1,2} = \langle x_1 \wedge \bar{x}_1 \wedge \bar{y}_2, x_1 \wedge \bar{y}_1 \wedge \bar{y}_3, y_1 \wedge \bar{x}_1 \wedge \bar{y}_1, y_1 \wedge \bar{x}_1 \wedge \bar{y}_3, y_1 \wedge \bar{y}_1 \wedge \bar{y}_2, y_1 \wedge \bar{y}_2 \wedge \bar{y}_3, y_2 \wedge \bar{x}_1 \wedge \bar{y}_2, y_2 \wedge \bar{y}_1 \wedge \bar{y}_3, y_3 \wedge \bar{x}_1 \wedge \bar{y}_1, y_3 \wedge \bar{x}_1 \wedge \bar{y}_3, y_3 \wedge \bar{y}_1 \wedge \bar{y}_2, y_3 \wedge \bar{y}_2 \wedge \bar{y}_3 \rangle.$$

We compute

$$H_{\bar{\partial}}^{1,0}(G/\Gamma) = \langle [x_1] \rangle, \quad H_{\bar{\partial}}^{0,1}(G/\Gamma) = \langle [\bar{x}_1], [\bar{y}_2] \rangle,$$

$$H_{\bar{\partial}}^{2,0} = 0, \quad H_{\bar{\partial}}^{1,1} = \langle [x_1 \wedge \bar{y}_2], [y_1 \wedge \bar{y}_3], [y_3 \wedge \bar{y}_1] \rangle, \quad H_{\bar{\partial}}^{0,2} = \langle [\bar{y}_1 \wedge \bar{y}_3] \rangle,$$

$$H_{\bar{\partial}}^{3,0} = \langle [y_1 \wedge y_2 \wedge y_3] \rangle, \quad H_{\bar{\partial}}^{0,3} = \langle [\bar{x}_1 \wedge \bar{y}_1 \wedge \bar{y}_3], [\bar{y}_1 \wedge \bar{y}_2 \wedge \bar{y}_3] \rangle,$$

$$H_{\bar{\partial}}^{2,1} = \langle [x_1 \wedge y_1 \wedge \bar{y}_3], [x_1 \wedge y_3 \wedge \bar{y}_1], [y_1 \wedge y_2 \wedge \bar{y}_3], [y_2 \wedge y_3 \wedge \bar{y}_1] \rangle,$$

$$H_{\bar{\partial}}^{1,2} = \langle [x_1 \wedge \bar{x}_1 \wedge \bar{y}_2], [x_1 \wedge \bar{y}_1 \wedge \bar{y}_3], [y_1 \wedge \bar{x}_1 \wedge \bar{y}_3], [y_1 \wedge \bar{y}_2 \wedge \bar{y}_3], [y_3 \wedge \bar{x}_1 \wedge \bar{y}_1], [y_3 \wedge \bar{y}_1 \wedge \bar{y}_2] \rangle.$$

We also compute

$$\begin{aligned}
H^1(G/\Gamma) &= \langle [x_1], [\bar{x}_1], [y_2 + \bar{y}_2] \rangle, \\
H^2(G/\Gamma) &= \langle [x_1 \wedge \bar{x}_1], [y_1 \wedge \bar{y}_3], [y_2 \wedge \bar{y}_2 - \bar{y}_1 \wedge \bar{y}_3 + y_1 \wedge y_3], [y_3 \wedge \bar{y}_1] \rangle, \\
H^3(G/\Gamma) &= \langle [y_1 \wedge y_2 \wedge y_3], [x_1 \wedge y_1 \wedge \bar{y}_3], [x_1 \wedge y_3 \wedge \bar{y}_1], [y_1 \wedge y_2 \wedge \bar{y}_3], [y_2 \wedge y_3 \wedge \bar{y}_1], \\
&\quad [y_3 \wedge \bar{x}_1 \wedge \bar{y}_1], [-x_1 \wedge y_2 \wedge \bar{x}_1 + x_1 \wedge \bar{x}_1 \wedge \bar{y}_2], [y_1 \wedge \bar{x}_1 \wedge \bar{y}_3], \\
&\quad [x_1 \wedge y_2 \wedge \bar{y}_2 - x_1 \wedge \bar{y}_1 \wedge \bar{y}_3 + x_1 \wedge y_1 \wedge y_3], [y_1 \wedge \bar{y}_2 \wedge \bar{y}_3] \\
&\quad [-y_2 \wedge \bar{x}_1 \wedge \bar{y}_2 - \bar{x}_1 \wedge \bar{y}_1 \wedge \bar{y}_1 + y_1 \wedge y_3 \wedge \bar{x}_1], [y_3 \wedge \bar{y}_1 \wedge \bar{y}_2], [\bar{y}_1 \wedge \bar{y}_2 \wedge \bar{y}_3] \rangle.
\end{aligned}$$

By these computations, we have

$$\begin{aligned}
\dim H_{\bar{\partial}}^{1,0}(G/\Gamma) + \dim H_{\bar{\partial}}^{0,1}(G/\Gamma) &= 3 = \dim H^1(G/\Gamma), \\
\dim H_{\bar{\partial}}^{2,0}(G/\Gamma) + \dim H_{\bar{\partial}}^{1,1}(G/\Gamma) + \dim H_{\bar{\partial}}^{0,2}(G/\Gamma) &= 4 = \dim H^2(G/\Gamma), \\
\dim H_{\bar{\partial}}^{3,0}(G/\Gamma) + \dim H_{\bar{\partial}}^{2,1}(G/\Gamma) + \dim H_{\bar{\partial}}^{1,2}(G/\Gamma) + \dim H_{\bar{\partial}}^{0,3}(G/\Gamma) &= 13 = \dim H^3(G/\Gamma).
\end{aligned}$$

Hence the proposition follows.  $\square$

#### REFERENCES

- [1] L. A. Cordero, M. Fernández, A. Gray, The Frölicher spectral sequence for compact nilmanifolds. *Illinois J. Math.* **35** (1991), no. 1, 56–67.
- [2] L. A. Cordero, M. Fernández, A. Gray, L. Ugarte, A general description of the terms in the Frölicher spectral sequence, *Differential Geom. Appl.* **7** (1997), 75–84.
- [3] K. Dekimpe, Semi-simple splittings for solvable Lie groups and polynomial structures. *Forum Math.* **12** (2000), no. 1, 77–96.
- [4] H. Kasuya, Techniques of computations of Dolbeault cohomology of solvmanifolds. *Math. Z.* DOI: 10.1007/s00209-012-1013-0 (online first). arXiv:1107.4761 (preprint)
- [5] H. Kasuya, Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems. To appear in *J. Differential Geometry*. <http://arxiv.org/abs/1009.1940>.
- [6] H. Kasuya, Hodge symmetry and decomposition on non-Kähler solvmanifolds. <http://arxiv.org/abs/1109.5929>
- [7] H. Kasuya, de Rham and Dolbeault Cohomology of solvmanifolds with local systems. arXiv:1207.3988v3
- [8] J. McCleary, *A user's guide to spectral sequences*, Second edition, Cambridge Studies in Advanced Mathematics, **58**, Cambridge University Press, Cambridge, 2001.
- [9] S. Rollenske, The Frölicher spectral sequence can be arbitrarily non-degenerate. *Math. Ann.* **341** (2008), no. 3, 623–628.
- [10] C. Voisin, *Hodge Theory and complex algebraic geometry I*, Cambridge studies in advanced mathematics, 76, Cambridge University Press 2002.

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